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Giant magnons in $AdS_4 \times CP^3$: embeddings, charges and a Hamiltonian

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Abstract: This paper studies giant magnons in CP^3, which in all known cases are old solutions from S^5 placed into two- and three-dimensional subspaces of CP^3, namely CP^1, RP^2 and RP^3. We clarify some points about these subspaces, and other potentially interesting three- and four-dimensional subspaces. After confirming that $\Delta - (J_1 - J_4)/2$ is a Hamiltonian for small fluctuations of the relevant 'vacuum' point particle solution, we use it to calculate the dispersion relation of each of the inequivalent giant magnons. We comment on the embedding of finite-J solutions, and use these to compare string solutions to giant magnons in the algebraic curve.

Keywords: AdS-CFT Correspondence, Bosonic Strings

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Contents

1 Introduction

Classical string solutions in $AdS_5 \times S^5$ have played an important role in the study of the duality to $\mathcal{N} = 4$ SYM [1–3]. It seems that this pattern is being repeated in the new $\mathcal{N} = 6$ duality [4], in which planar superconformal Chern-Simons theory is dual to string theory on $AdS_4 \times CP^3$. Some of the most interesting recent papers study strings moving in an $AdS_2 \times S^1$ subspace, where although the classical solutions are identical to those long used in the $\mathcal{N} = 4$ case, the quantum properties are different. The results from semiclassical quantisation [5–8] can be compared to those from the asymptotic Bethe ansatz, and at present there appear to be some difficulties [9].

This paper is instead about string solutions exploring primarily the CP^3 factor. One would expect to find analogues of the giant magnons [3] here, which in the $\mathcal{N} = 4$ case live in an $S^2 \subset S^5$. And indeed, it turns out that the same solutions exist in CP^3 [10, 11]. There are two inequivalent ways to embed the basic S^2 magnon, into either $CP^1 = S^2$ or $RP^2 = S^2/\mathbb{Z}_2$, [10] both two-dimensional subspaces of CP^3.

In either theory, the anomalous dimension can be calculated as the Hamiltonian of some spin chain [10, 12–14]. The giant magnons are dual to the elementary excitations of this spin chain, and have a periodic dispersion relation $\Delta - J = \sqrt{1 + f^2(\lambda)\sin^2(p/2)}$ which on the gauge side is an symptom of the discrete spatial dimension of the spin chain, and on the string side arises from p being an angle along an equator. The conformal dimension Δ and the R-charge J are mapped by AdS/CFT to energy and angular momentum of the string state. For the state dual to the (ferromagnetic) vacuum of the spin chain, which is a point particle, $\Delta - J$ becomes the Hamiltonian for small fluctuations. We confirm that in the $\mathcal{N} = 6$ case, the difference $\Delta - (J_1 - J_4)/2$ has the same property.

An important difference between the old $\mathcal{N} = 4$ case and the new $\mathcal{N} = 6$ case is the behaviour of the function $f(\lambda)$, the only part of the dispersion relation not fixed by supersymmetry [3, 15]. In the old case, calculations of $f(\lambda)$ at both large and small λ give $f(\lambda) = \sqrt{\lambda}/\pi$, and this is conjectured to be true for all λ. In the new case, however, the function (often called h instead) is $h(\lambda) = \lambda$ at small λ but $h(\lambda) \sim \lambda^{1/2}$ at large λ. Our knowledge of this function at large λ comes (in both cases) from studying classical string theory, and so depends on the correct identification of the relevant string solutions.

Dyonic giant magnons are those with more than one large angular momentum, dual to a large condensate of impurities on the spin chain. These are string solutions in S^3, and they can at least sometimes be embedded into CP^3 in much the same way as the basic magnon, generalising the RP^2 magnons and living in an RP^3 subspace [16, 17]. There is room for dyonic solutions with other angular momenta, truly exploring CP^3, including those generalising the CP^1 magnon. While we have not been able to find such solutions, we discuss where they might live. The subspace frequently called $S^2 \times S^2$ in the literature is in fact just RP^2, and while there is a genuine $S^2 \times S^2$ subspace, one cannot place arbitrary S^2 string solutions into each factor, because the equations of motion couple the two factors. Likewise the $S^2 \times S^1$ subspace studied by [17] has extra constraints limiting what solutions can exist there.

Outline. In section 2 we write down a few relevant facts about ABJM theory and its spin-chain description, and in section 3 we look at its string dual in $AdS_4 \times CP^3$. In section 4 we calculate fluctuations about the point particle solution corresponding to the spin chain vacuum, showing that $\Delta - (J_1 - J_4)/2$ is a Hamiltonian for these.

Section 5 is a catalogue of existing giant magnon solutions in various subspaces of CP^3: single-spin magnons in CP^1 and RP^2, and dyonic magnons in RP^3. Section 6 looks at other subspaces of potential interest, including the four-dimensional spaces $S^2 \times S^2$ and

CP^2, and also $S^2 \times S^1$. Section 7 is a brief discussion of finite-J solutions, which can be embedded in the same way, and their dispersion relations.

We discuss and conclude in section 8. Extra details of the geometry, and how to analyse strings in it using Lagrange multipliers, are discussed in two appendices.

Note added in proof. After this paper's appearance on the archive, but before its appearance in the journal, two new string solutions not known in S^5 have appeared, thus the first sentence of this paper's abstract is no longer true. One was found by [18–20], and another by [21].

2 Groups in ABJM theory

The $\mathcal{N} = 6$ superconformal Chern-Simons-matter theory[1]of ABJM [4] of interest here has gauge symmetry $\mathrm{U}(N) \times \mathrm{U}(N)$. We will only study its scalars A_i, B_i. The fields A_1, A_2 are matrices in the $(N, \bar{N})$ representation of this (one fundamental index, one anti-fundamental), and the fields B_1, B_2 in the $(\bar{N}, N)$. There is a manifest $\mathrm{SU}(2)_A$ R-symmetry in which the As form a doublet, and $\mathrm{SU}(2)_B$ acting on the Bs. There is also the conformal group $\mathrm{SO}(2,3)$, since we are in 2+1 dimensions. Taking spacetime to be $\mathbb{R} \times S^2$, we restrict attention to fields in the lowest Kaluza-Klein mode on this S^2, i.e. in the singlet representation of $\mathrm{SO}(3)_r$, which is the spatial part of the conformal group.

In [24] it was proven that the full R-symmetry is in fact SU(4), with the following vector in the fundamental representation:

$$Y^A = (A_1, A_2, B_1^\dagger, B_2^\dagger) \tag{2.1}$$

and $Y_A^\dagger$ in the anti-fundamental. If we keep only $(Y^1, Y^4) = (A_1, B_2^\dagger)$ then we have a subgroup called $\mathrm{SU}(2)_{G'}$, and if we keep only $(Y^2, Y^3) = (A_2, B_1^\dagger)$ then we have the subgroup $\mathrm{SU}(2)_G$.[2]

This theory is dual to membranes on $AdS_4 \times S^7/\mathbb{Z}_k$, where $(k, -k)$ are the level numbers of the two Chern-Simons terms. The 't Hooft limit $N \to \infty$ with $\lambda = N/k$ fixed sends $k \to \infty$, and reduces the dual theory to type IIA strings on $AdS_4 \times CP^3$.

To find a spin-chain description, [10, 13, 14] study gauge invariant operators of length $2L$ of the form

$$\mathcal{O} = \chi^{B_1 B_2 \cdots B_L}_{A_1 A_2 \cdots A_L} \operatorname{tr} Y^{A_1} Y^\dagger_{B_1}\, Y^{A_2} Y^\dagger_{B_2}\, \ldots\, Y^{A_L} Y^\dagger_{B_L}.$$

When χ is fully symmetric (in the As, and in the Bs) and traceless, $\mathcal{O}$ is a chiral primary, thus protected, and has scaling dimension $\Delta = L$. In this case the anomalous dimension, defined $D = \Delta - L$, will be zero.

The $\mathrm{SU}(2) \times \mathrm{SU}(2)$ sector refers to operators $\mathcal{O}$ in which only Y^1, Y^2 and $Y_3^\dagger$, $Y_4^\dagger$ appear. (That is, only fields A_1, A_2, B_1 and B_2. The two factors in the name are $\mathrm{SU}(2)_A$

[1]These of theories were discovered after the explorations of 3-dimensional superconformal theories with non-Lie-algebra guage symmetry by BLG, [22] and build on earlier work on Chern-Simons-matter theories by [23].

[2]These subscripts are the notation of [10], except that they have B_1 and B_2 the other way around: their spin chain vacuum is $\operatorname{tr}(A_1 B_1^\dagger)^L$ rather than the $\operatorname{tr}(Y^1 Y_4^\dagger)^L$ of [13] which we use, (2.2).

and $\mathrm{SU}(2)_B$). The SU(3) sector allows operators with Y^1, Y^2, Y^3 and $Y_4^\dagger$. For both of these, the vacuum is taken to be

$$\mathcal{O}_{\rm vac} = \mathrm{tr}\,\left(Y^1 Y_4^\dagger\right)^L. \tag{2.2}$$

This has $\Delta = L$, and $J = L$, where J is the Cartan generator in $\mathrm{SU}(2)_{G'}$: $J(Y^1) = \frac{1}{2}$ and $J(Y^4) = -\frac{1}{2}$, thus $J(Y_4^\dagger) = +\frac{1}{2}$.

In the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ sector, the two-loop anomalous scaling dimension is computed by the sum of the Hamiltonians of two independent Heisenberg XXX spin chains, for the even and odd sites. The momentum constraint (from the $\mathrm{U}(N)$ trace tr) is that the sum of their momenta be zero. (This is slightly weaker than the $\mathcal{N} = 4$ case, [12] where there is one total momentum which must be zero).

3 The geometry of CP^3

The string dual of ABJM theory (in the 't Hooft limit) lives in the 10-dimensional space $AdS_4 \times CP^3$, with sizes specified by the metric

$$ds^2 = \frac{R^2}{4} ds^2_{AdS_4} + R^2 ds^2_{CP^3} \tag{3.1}$$

where $R^2 = 2^{5/2}\pi\sqrt{\lambda}$. The large-$\lambda$ limit gives strongly coupled gauge theory, dual to classical strings. In addition to this (string-frame) metric, there is a dilaton and RR forms, given by [4], which do not influence the motion of classical strings.

The metric for CP^3 is given in [4] as

$$ds^2_{CP^3} = \frac{dz_i d\bar{z}_i}{\rho^2} - \frac{|z_i d\bar{z}_i|^2}{\rho^4}, \qquad \text{where } \rho^2 = z_i \bar{z}_i \tag{3.2}$$

in terms of the homogeneous co-ordinates $\mathbf{z} \in \mathbb{C}^4$, where $\mathbf{z} \sim \lambda\mathbf{z}$ for any complex λ. The SU(4) isometry symmetry is manifest here, with $\mathbf{z}$ in the fundamental representation. AdS/CFT identifies this isometry group with the SU(4) R-symmetry group, so it is natural to take $\mathbf{z}$ to be in the same basis as the fields Y^A in (2.1) above.

There are two angular parameterisations commonly used. One set of angles was given by [25]:

$$\begin{aligned} ds^2_{CP^3} = {} & d\mu^2 + \frac{1}{4}\sin^2\mu\cos^2\mu\left[d\chi + \sin^2\alpha\ (d\psi + \cos\theta\, d\phi)\right]^2 \\ & + \sin^2\mu\left[d\alpha^2 + \frac{1}{4}\sin^2\alpha\left(d\theta^2 + \sin^2\theta\, d\phi^2 + \cos^2\alpha\,(d\psi + \cos\theta\, d\phi)^2\right)\right] \end{aligned} \tag{3.3}$$

with ranges $\alpha, \mu \in [0, \frac{\pi}{2}]$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$ and $\psi, \chi \in [0, 4\pi]$. Another was given by [26]:

$$\begin{aligned} ds^2_{CP^3} = {} & d\xi^2 + \frac{1}{4}\sin^2 2\xi\left(d\eta + \frac{1}{2}\cos\vartheta_1\, d\varphi_1 - \frac{1}{2}\cos\vartheta_2\, d\varphi_2\right)^2 \\ & + \frac{1}{4}\cos^2\xi\left(d\vartheta_1^2 + \sin^2\vartheta_1\, d\varphi_1^2\right) + \frac{1}{4}\sin^2\xi\left(d\vartheta_2^2 + \sin^2\vartheta_2\, d\varphi_2^2\right) \end{aligned} \tag{3.4}$$

where $\xi \in [0, \frac{\pi}{2}]$, $\vartheta_1, \vartheta_2 \in [0, \pi]$, $\varphi_1, \varphi_2 \in [0, 2\pi]$ and $\eta \in [0, 4\pi]$. (This can be obtained by building S^7 from $S^3 \times S^3$ with the seventh co-ordiante ξ controlling their relative sizes). In appendix A we give the maps between these angles and the homogeneous co-ordinates.

The Penrose limit describes the geometry very near to a null geodesic [27] and has been very important in AdS/CFT [28]. This has been studied in $AdS_4 \times CP^3$ by [10], where the particle travels along $\chi = 4t$ with $\alpha = 0$, $\mu = \pi/4$ in terms of the angles in (3.3), and by [11, 29], who use co-ordinates (3.4), expanding near $\vartheta_1 = \vartheta_2 = 0$, $\xi = \pi/4$ with distance along the line $\tilde{\psi} = \eta + (\varphi_1 - \varphi_2)/2 = -2t$. In all cases, the test particle moves along the path[3]

$$\mathbf{z} = \frac{1}{\sqrt{2}} \left(e^{it},\, 0,\, 0,\, e^{-it} \right). \tag{3.5}$$

This has large angular momentum in opposite directions on the z_1 and z_4 planes, as one would expect for the state dual to the operator (2.2). This led [13] to write this state down as the string state dual to the vacuum $\mathcal{O}_{\text{vac}}$.

4 Fluctuation Hamiltonian for the point particle

In the $AdS_5 \times S^5$ case, the string state dual to the spin chain vacuum $\text{tr}(\Phi_1 + i\Phi_2)^L$ is a point particle with $X = (\cos t, \sin t,\ 0, 0,\ 0, 0)$. This state has large angular momentum in the 1-2 plane, $J = \Delta$. By studying small fluctuations of this state, viewed as a string solution, one can show that $\Delta - J$ is a Hamiltonian for the physical modes [2]. Semiclassical quantisation treats these modes as quantum fields with energy $\Delta - J$. Giant magnons are exitations above this vacuum, and so their semiclassical quantisation involves calculating quantum corrections to this energy [30].

In the present $AdS_4 \times CP^3$ case, given the point particle state (3.5) and the vacuum (2.2), it is reasonable to guess that $\Delta - (J_1 - J_4)/2$ will play the same role. Here we confirm this, by explicitly deriving the fluctuation Hamiltonian.

Write the metric for the AdS_4 factor in the form

$$ds^2_{AdS_4} = -\left(\frac{1 + \mathbf{r}^2}{1 - \mathbf{r}^2} \right)^2 d\tau^2 + \frac{4}{(1 - \mathbf{r}^2)^2} d\mathbf{r}^2 \tag{4.1}$$

where $\mathbf{r} = r_i$, $i = 1, 2, 3$ are zero at the centre of AdS, and τ is AdS time. (In our notation worldsheet space and time are x, t). For the CP^3 sector we use yet another set of co-ordinates, which are convenient for this calculation.[4] We write

$$\mathbf{z} = \left(e^{i\beta} \frac{1 + \epsilon}{\sqrt{2}},\ y_1 + i y_2,\ y_3 + i y_4,\ e^{-i\beta} \frac{1 - \epsilon}{\sqrt{2}} \right) \tag{4.2}$$

[3] We stress that there are not different Penrose limits for the different giant magnon sectors. To get precisely this path $\mathbf{z}$, using our conventions given in (A.2) and (A.3), we fix in addition $\theta = \pi$ (in the first case) and $\varphi_1 = \varphi_2$ (in the second), and also swop $z_2 \leftrightarrow z_4$ in the second case.

[4] The advantage of these co-ordinates (as opposed to the angles) is that the identification of the charges J_i here with those for the magnons in section 5 and those for the gauge theory in section 2 is transparent. To cover the whole space with these co-ordinates we need $\beta \in [0, \pi]$ and $\epsilon \in [-1, 1)$. This is clearly seen in terms of the inhomogeneous co-ordinates $z_1/z_4 = e^{i2\beta}(1 + \epsilon)/(1 - \epsilon)$ and z_2/z_4, z_3/z_4. (Similar, but not identical, co-ordinates were used by [6]).

in terms of which $\rho^2 = \bar{z}_i z_i = 1 + \epsilon^2 + \mathbf{y}^2$ (where $\mathbf{y}^2 = y_j y_j$). The metric (3.2) then becomes

$$\begin{aligned} ds^2_{CP^3} &= \frac{(1+\epsilon^2)d\beta^2 + d\epsilon^2 + d\mathbf{y}^2}{1+\epsilon^2+\mathbf{y}^2} \\ &\quad - \frac{(\epsilon d\epsilon + \mathbf{y}\cdot d\mathbf{y})^2 + (2\epsilon d\beta + y_1 dy_2 - y_2 dy_1 + y_3 dy_4 - y_4 dy_3)^2}{(1+\epsilon^2+\mathbf{y}^2)^2}. \end{aligned}$$

Putting these together, and dropping R^2 in (3.1) (because we pull it out to be the action's prefactor) the full metric becomes

$$\begin{aligned} ds^2 &= \frac{1}{4} ds^2_{AdS_4} + ds^2_{CP^3} \\ &= \left(-\frac{1}{4} - \mathbf{r}^2\right) d\tau^2 + d\mathbf{r}^2 + (1 - 4\epsilon^2 - \mathbf{y}^2) d\beta^2 + d\epsilon^2 + d\mathbf{y}^2 + \cdots . \end{aligned} \tag{4.3}$$

On the second line here we expand near $\mathbf{r} = \mathbf{y} = 0$, $\epsilon = 0$ and present only the terms that we will need. The point particle travels on the line $\tau = 2t$, $\beta = t$, and we define perturbations about this as follows:

$$\begin{aligned} \tau &= 2t + \frac{1}{\lambda^{1/4}}\tilde{\tau} & \mathbf{r} &= \frac{1}{\lambda^{1/4}}\tilde{\mathbf{r}} \\ \beta &= t + \frac{1}{\lambda^{1/4}}\tilde{\beta} & \epsilon &= \frac{1}{\lambda^{1/4}}\tilde{\epsilon} \\ & & \mathbf{y} &= \frac{1}{\lambda^{1/4}}\tilde{\mathbf{y}} . \end{aligned} \tag{4.4}$$

The perturbations $\tilde{\tau}$ and $\tilde{\beta}$ will lead to modes which are pure gauge, but are needed for now to maintain conformal gauge.

The Lagrangian is $\mathcal{L} = \frac{1}{2}(-\gamma_{00} + \gamma_{11})$ and the Virasoro constraints are $\gamma_{00} + \gamma_{11} = 0$ and $\gamma_{01} = 0$, in terms of the induced metric γ_{ab}. The components we need are:

$$\begin{aligned} \gamma_{00} &= G_{\mu\nu}\partial_t X^\mu \partial_t X^\nu \\ &= \frac{1}{\lambda^{1/4}}\left[-\partial_t\tilde{\tau} + 2\partial_t\tilde{\beta}\right] \\ &\quad + \frac{1}{\sqrt{\lambda}}\left[-\frac{(\partial_t\tilde{\tau})^2}{4} + (\partial_t\tilde{\mathbf{r}})^2 + (\partial_t\tilde{\beta})^2 + (\partial_t\tilde{\epsilon})^2 + (\partial_t\tilde{\mathbf{y}})^2 - 4\tilde{\mathbf{r}}^2 - 4\tilde{\epsilon}^2 - \tilde{\mathbf{y}}^2\right] \\ &\quad + \frac{1}{\lambda^{3/4}}\left[-4\tilde{\mathbf{r}}^2\partial_t\tilde{\tau} + \partial_t\tilde{\beta}\,(\ldots) + \partial_t\tilde{\mathbf{y}}\cdot(\ldots)\right] + o\left(\frac{1}{\lambda}\right) \end{aligned}$$

where $(\ldots)$ indicates terms not needed for this calculation, and

$$\begin{aligned} \gamma_{11} &= G_{\mu\nu}\partial_x X^\mu \partial_x X^\nu \\ &= \frac{1}{\sqrt{\lambda}}\left[-\frac{(\partial_x\tilde{\tau})^2}{4} + (\partial_x\tilde{\mathbf{r}})^2 + (\partial_x\tilde{\beta})^2 + (\partial_x\tilde{\epsilon})^2 + (\partial_x\tilde{\mathbf{y}})^2\right] + o\left(\frac{1}{\lambda}\right). \end{aligned}$$

Next we define the string's conserved charges. Δ is the charge generated by time translation:

$$\begin{aligned} \Delta &= 2\sqrt{2\lambda}\int dx\, \frac{\partial\mathcal{L}\left[\tau, \mathbf{r}, \beta, \epsilon, \mathbf{y}\right]}{\partial\,\partial_t\tau} \\ &= 2\sqrt{2}\,\lambda^{3/4}\int dx\, \frac{\partial\tilde{\mathcal{L}}\left[\tilde{\tau}, \tilde{\mathbf{r}}, \tilde{\beta}, \tilde{\epsilon}, \tilde{\mathbf{y}}\right]}{\partial\,\partial_t\tilde{\tau}} \end{aligned}$$

and J_i is the charge generated by rotation of the z_i complex plane:[5]

$$\begin{aligned} J_1 &= 2\sqrt{2\lambda}\int dx\, \frac{\partial\mathcal{L}}{\partial\,\partial_t(\arg Z_1)} \\ &= 2\sqrt{2\lambda}\int dx \left[\frac{\mathrm{Im}\left(\bar{Z}_1\partial_t Z_1\right)}{\rho^2} - \frac{|Z_1|^2\sum_i \mathrm{Im}\left(\bar{Z}_i\partial_t Z_i\right)}{\rho^4}\right] \qquad (4.5) \\ J_4 &= 2\sqrt{2\lambda}\int dx \left[\frac{\mathrm{Im}\left(\bar{Z}_4\partial_t Z_4\right)}{\rho^2} - \frac{|Z_4|^2\sum_i \mathrm{Im}\left(\bar{Z}_i\partial_t Z_i\right)}{\rho^4}\right]. \end{aligned}$$

Substituting in the above mode definitions, we get

$$\begin{aligned} \Delta &= \sqrt{2}\int dx\left[\sqrt{\lambda} + \frac{\lambda^{1/4}}{2}\partial_t\tilde{\tau} + 4\tilde{\mathbf{r}}^2 + o\left(\frac{1}{\lambda^{1/4}}\right)\right] \qquad (4.6) \\ J_1 &= \sqrt{2}\int dx\left[\sqrt{\lambda} + \lambda^{1/4}\partial_t\tilde{\beta} - 4\tilde{\epsilon}^2 - \tilde{\mathbf{y}}^2 + (\tilde{y}_2\partial_t\tilde{y}_1 - \tilde{y}_1\partial_t\tilde{y}_2 + \tilde{y}_4\partial_t\tilde{y}_3 - \tilde{y}_3\partial_t\tilde{y}_4) + o\left(\frac{1}{\lambda^{1/4}}\right)\right] \\ J_4 &= \sqrt{2}\int dx\left[-\sqrt{\lambda} - \lambda^{1/4}\partial_t\tilde{\beta} + 4\tilde{\epsilon}^2 + \tilde{\mathbf{y}}^2 + (\tilde{y}_2\partial_t\tilde{y}_1 - \tilde{y}_1\partial_t\tilde{y}_2 + \tilde{y}_4\partial_t\tilde{y}_3 - \tilde{y}_3\partial_t\tilde{y}_4) + o\left(\frac{1}{\lambda^{1/4}}\right)\right]. \end{aligned}$$

These diverge as $\lambda \to \infty$, but for the linear combination used below, the $o(\sqrt{\lambda})$ terms cancel. The $o(\lambda^{1/4})$ terms, linear in the fluctuations, can be re-written as quadratic $o(1)$ terms using the Virasoro constraint $\gamma_{00} + \gamma_{11} = 0$. This leads to

$$\begin{aligned} \Delta - \frac{J_1 - J_4}{2} \\ = \frac{\sqrt{2}}{2}\int dx \Big[&(\partial_t\tilde{\mathbf{r}})^2 + (\partial_x\tilde{\mathbf{r}})^2 + 4\tilde{\mathbf{r}}^2 + (\partial_t\tilde{\epsilon})^2 + (\partial_x\tilde{\epsilon})^2 + 4\tilde{\epsilon}^2 + (\partial_t\tilde{\mathbf{y}})^2 + (\partial_x\tilde{\mathbf{y}})^2 + \tilde{\mathbf{y}}^2 \\ &- \frac{(\partial_t\tilde{\tau})^2}{4} - \frac{(\partial_x\tilde{\tau})^2}{4} + (\partial_t\tilde{\beta})^2 + (\partial_x\tilde{\beta})^2\Big] + o\left(\frac{1}{\lambda^{1/4}}\right). \end{aligned}$$

The terms on the last line are the gauge modes, generating infinitesimal reparameterisations, so would not be included in semiclassical quantisation. After dropping these, we are left with the Hamiltonian[6] $\Delta - \frac{J_1 - J_4}{2} = \sqrt{2}\int dx\,\mathcal{H}$, where[7]

$$\mathcal{H} = \frac{1}{2}\left[\,(\partial_t\tilde{\mathbf{r}})^2 + (\partial_x\tilde{\mathbf{r}})^2 + 4\tilde{\mathbf{r}}^2 \;+\; (\partial_t\tilde{\epsilon})^2 + (\partial_x\tilde{\epsilon})^2 + 4\tilde{\epsilon}^2 \;+\; (\partial_t\tilde{\mathbf{y}})^2 + (\partial_x\tilde{\mathbf{y}})^2 + \tilde{\mathbf{y}}^2\,\right].$$

[5] Note that in deriving these charges we treat $Z_1, \ldots, Z_4$ as independent fields, even though they are in fact related through $\mathbf{Z} \sim \lambda\mathbf{Z}$, which defines CP^3 from $\mathbb{C}^4$. Therefore, we do this before adopting the parametrisation (4.2), in which we have fixed some of this gauge freedom by writing only six (not eight) real co-ordinates.

[6] This $\mathcal{H}$ is the two-dimensional Hamiltonian that one would obtain from the quadratic part of the fluctuation Lagrangian $\mathcal{L} = \frac{1}{2}(-\gamma_{00} + \gamma_{11})$ by naively dropping terms linear in time derivative and reversing the signs of the terms quadratic in the time derivative. But note that without dropping these $o(\lambda^{1/4})$ terms, the string Hamiltonian is fixed to zero by the Virasoro constraint $\gamma_{00} + \gamma_{11} = 0$, which we have used to derive $\mathcal{H}$.

[7] The obvious charges one could add to $\Delta - (J_1 - J_4)/2$, while keeping it finite, are J_2 and J_3. These will add terms like $\tilde{y}_2\partial_t\tilde{y}_1 - \tilde{y}_1\partial_t\tilde{y}_2$ to $\mathcal{H}$.

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This describes eight massive modes: the three $\tilde{r}_i$ in AdS_4, plus $\tilde{\epsilon}$ and the four $\tilde{y}_i$ in CP^3. As was noted by [6], one of the CP^3 modes, $\tilde{\epsilon}$, has reached across the aisle to have the same mass as the AdS modes $\tilde{\mathbf{r}}$. The same list of masses was also found by [10, 11, 29] when studying the Penrose limit, and by [5, 6] for modes of spinning strings in the $AdS_2 \times S^1$ subspace.

5 Placing giant magnons into CP^3

Recall that the Hoffman-Maldacena giant magnon [3] is a rigidly rotating classical string solution in $\mathbb{R} \times S^2$, given in timelike conformal gauge by

$$\begin{aligned} \cos\theta_{\rm mag} &= \sin\frac{p}{2}\,\mathrm{sech}\,u \\ \tan\left(\phi_{\rm mag} - t\right) &= \tan\frac{p}{2}\tanh u \end{aligned} \tag{5.1}$$

where $u = (x - t\cos\frac{p}{2})/\sin\frac{p}{2}$ is the boosted spatial co-ordinate for a soliton with worldsheet velocity $\cos(p/2)$. The spacetime is $ds^2 = -dt^2 + d\theta^2 + \sin^2\theta\, d\phi^2$ — by timelike gauge we mean that the target-space time is also worldsheet time.[8]

We define conserved charges here as follows:

$$\Delta = \sqrt{2\lambda}\int dx\, 1 \tag{5.2}$$

$$J_{\rm sphere} = \sqrt{2\lambda}\int dx\ \mathrm{Im}\left(\bar{W}_1 \partial_t W_1\right). \tag{5.3}$$

This Δ matches (4.6) used above when the AdS fluctuations $\tilde{\tau}$ and $\tilde{\mathbf{r}}$ are turned off. Note that we keep the same prefactor $\sqrt{2\lambda}$ here, which is not the one we would use in the $AdS_5 \times S^5$ case. Finally, we write the complex embedding co-ordinates $W_1 = e^{i\phi_{\rm mag}}\sin\theta_{\rm mag}$ and $W_2 = \cos\theta_{\rm mag}$.[9]

Both Δ and $J_{\rm sphere}$ are infinite for the solution (5.1), but their difference is finite:

$$\Delta - J_{\rm sphere} = 2\sqrt{2\lambda}\,\sin\left(\frac{p}{2}\right).$$

The parameter p is the (absolute value of the) momentum of the spin chain excitation in the dual gauge theory, which is why this is called a dispersion relation. It is also equal to the opening angle $\Delta\phi_{\rm mag}$ of the string solution on the equator $\theta_{\rm mag} = \frac{\pi}{2}$.

We now turn to solutions in $\mathbb{R} \times CP^3$, with metric $ds^2 = -dt^2 + ds^2_{CP^3}$. All solutions will be in conformal gauge, and with worldsheet time t related to AdS time τ by $\tau = 2t$, so we will continue to use the definition of Δ from (5.2), although for J we must now use (4.5). We will also continue to use the parameter $p \in [0, 2\pi]$ in all the cases below, and while this should still be a momentum in the dual theory, we make no comment here on the precise factors involved.

[8] What we call timelike conformal gauge is sometimes called static conformal gauge. In our conventions, AdS time τ is given by $\tau = 2t$. However, because of the factor $\frac{1}{4}$ in the metric (4.3), it is t rather than τ which is physical time.

[9] Our notation is that (w_1, w_2) are complex embedding co-ordinates for the sphere, while z_i are for CP^3. Capital letters indicate a string solution in this space.

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5.1 The subspace CP^1

If we set $z_2 = z_3 = 0$, or in terms of angles (3.3), $\alpha = 0$, then we obtain the space $CP^1 = S^2$ with metric

$$ds^2 = \frac{1}{4}\left[d(2\mu)^2 + \sin^2(2\mu)d\Big(\frac{\chi}{2}\Big)^2\right]. \tag{5.4}$$

This is a sphere of radius $\frac{1}{2}$, so to place the magnon solution (5.1) here (as was done by [10]) maintaining conformal gauge we need to set

$$\begin{aligned} 2\mu &= \theta_{\rm mag}(2x, 2t) \\ \frac{\chi}{2} &= \phi_{\rm mag}(2x, 2t)\,. \end{aligned} \tag{5.5}$$

Using the map (A.3), given in appendix A, and choosing $\theta = \pi$, we obtain

$$\begin{aligned} \mathbf{Z}(x,t) &= \frac{1}{\sqrt{2}}\left(e^{\frac{i}{2}\phi_{\rm mag}(2x,2t)}\sqrt{1-\cos\theta_{\rm mag}(2x,2t)}\,,\ 0,\ 0,\ e^{-\frac{i}{2}\phi_{\rm mag}}\sqrt{1+\cos\theta_{\rm mag}}\right) \\ &= \left(e^{it+f(2u)}\sin\frac{\theta_{\rm mag}(2x,2t)}{2}\,,\ 0,\ 0,\ e^{-it-f(2u)}\cos\frac{\theta_{\rm mag}(2x,2t)}{2}\right). \end{aligned} \tag{5.6}$$

Calculating charges for this solution, using definitions (4.5) for J and (5.2) for Δ, we recover the dispersion relation[10]

$$\Delta - \frac{J_1 - J_4}{2} = \sqrt{2\lambda}\,\sin\left(\frac{p}{2}\right). \tag{5.7}$$

We should check that this subspace is a legal one, meaning that solutions found here are guaranteed to be solutions in the full space. This can be done by finding the conformal gauge equations of motion coming from the Polyakov action with the metric (3.3), and confirming that α's equation is solved by $\alpha = 0$.[11] But in this case it is easier to note that $z_2 = z_3 = 0$ trivially solves their equations of motion, (B.2), which we derive in appendix B.

5.2 The subspace RP^2

A second embedding of the S^2 solution was first used by [11][12]

$$\mathbf{Z}(x,t) = \frac{1}{\sqrt{2}}\left(e^{i\phi_{\rm mag}(x,t)}\sin\theta_{\rm mag}(x,t)\,,\ \cos\theta_{\rm mag}\,,\ \cos\theta_{\rm mag}\,,\ e^{-i\phi_{\rm mag}}\sin\theta_{\rm mag}\right). \tag{5.8}$$

[10] Note that if you were to omit the second term in (4.5) when calculating J, thus effectivly using (5.3) appropriate for the sphere, you would get instead $\Delta - (J_1 - J_4)/2 = \sqrt{2\lambda}\,p\cos\left(\frac{p}{2}\right)$. In the RP^2 and RP^3 subspaces discussed below, this second term vanishes.

[11] In addition to solving the conformal gauge equations of motion, a string solution must be in conformal gauge, i.e. must solve the Virasoro constraints. If the solution on the subspace is in conformal gauge, then it follows trivially that the solution in the full space is too: the induced metric $\gamma_{ab} = \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$ is influenced only by those directions the solution explores, and in these directions the metric $G_{\mu\nu}$ is the same in both the full space and the subspace.

[12] We discuss the equations of motion used by [11] for strings in CP^3 in appendix B.2.

This solution lives in an RP^2 subspace, as can be seen by simply rotating some of the planes in $\mathbb{C}^4 = \mathbb{R}^8$ by $\frac{\pi}{4}$: in terms of new co-ordinates $\mathbf{w}$ defined by

$$\begin{aligned} w_1 &= \frac{1}{\sqrt{2}}(z_1 + \bar{z}_4) \qquad & w_4 &= \frac{1}{\sqrt{2}}(z_1 - \bar{z}_4) \\ w_2 &= \frac{1}{\sqrt{2}}(z_2 + \bar{z}_3) \qquad & w_3 &= \frac{1}{\sqrt{2}}(z_2 - \bar{z}_3)\,, \end{aligned} \tag{5.9}$$

this solution has $w_3 = w_4 = 0$ and is precisely the original giant magnon in the other two co-ordinates:

$$(W_1, W_2) = \left(e^{i\phi_{\rm mag}} \sin\theta_{\rm mag}\,,\ \cos\theta_{\rm mag}\right).$$

The reason this is RP^2 rather than S^2 is that sending $(w_1, w_2) \to -(w_1, w_2)$ gives an overall sign change on $\mathbf{z}$, and these two points are identified in CP^3.[13]

The subspace which this magnon explores can also be obtained from the metric (3.4), by fixing $\vartheta_1 = \frac{\pi}{2}$, $\vartheta_2 = \frac{\pi}{2}$, $\varphi_1 = 0$ and $\eta = 0$. The metric then becomes

$$ds^2 = d\xi^2 + \sin^2\xi \, d\Big(\frac{\varphi_2}{2}\Big)^2$$

and the magnon (5.8) is simply $\xi = \theta_{\rm mag}(x,t)$, $\varphi_2 = 2\phi_{\rm mag}(x,t)$. This can be checked to be a legal restriction from the equations of motion for the four angles fixed.

This subspace is sometimes, rather misleadingly, referred to as $S^2 \times S^2$. It is true that $|z_1|^2 + |z_2|^2 = \frac{1}{2}$ and $|z_3|^2 + |z_4|^2 = \frac{1}{2}$, and $\operatorname{Im} z_2 = 0 = \operatorname{Im} z_3$. These restrictions alone would describe a subspace of $\mathbb{C}^4$, namely $S^2 \times S^2 \subset \mathbb{C}^2 \times \mathbb{C}^2$. But we are in CP^3, not $\mathbb{C}^4$, and the space described by θ, ϕ (or by ξ, φ_2) has only two dimensions — these two S^2 factors are not independent. In section 6.2 below we discuss a genuine four-dimensional $S^2 \times S^2$ subspace.

The charges of this solution are very simply related to those of the magnon on the sphere, since the extra term in the CP^3 angular momentum (4.5) compared to the that for the sphere vanishes: $J_{\rm sphere} = J_1 = \frac{1}{2}(J_1 - J_4)$, and we get simply

$$\Delta - \frac{J_1 - J_4}{2} = 2\sqrt{2\lambda}\,\sin\left(\frac{p}{2}\right). \tag{5.10}$$

One difference from the magnon on S^2 is that when $p = \pi$, the magnon becomes a single closed string. Its cusps, at opposite points on the equator of S^2, are in fact at the same point in RP^2. In general the magnon connects two points a distance $\Delta\varphi_2 = 2\Delta\phi_{\rm mag} = 2p$ apart on the equator, but $\varphi \sim \varphi + 2\pi$ so $p = \delta$ and $p = \pi + \delta$ both connect the same two points. As was noted by [10], this can be viewed as giving rise to a second class of magnons, with

$$\Delta - \frac{J_1 - J_4}{2} = 2\sqrt{2\lambda}\,\sin\left(\frac{\pi + \delta}{2}\right) = 2\sqrt{2\lambda}\,\cos\left(\frac{\delta}{2}\right).$$

[13] In S^2, the standard co-ordinates have ranges $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$, and changing $\theta \to \pi - \theta$ and $\phi \to \phi + \pi$ simultaneously moves you to the antipodal point on S^2. But performing this change in the subspace of CP^3 parameterised by (5.8) changes $\mathbf{z} \to -\mathbf{z}$, and these two points are identified by the definition of CP^3. This is what makes the subspace $RP^2 = S^2/\mathbb{Z}_2$ instead of S^2. To obtain co-ordinates which cover this subspace only once, we can shorten the range of either θ or ϕ, and in figure 1 we choose to restrict to $\phi \in [0, \pi]$ while keeping $\theta \in [0, \pi]$.

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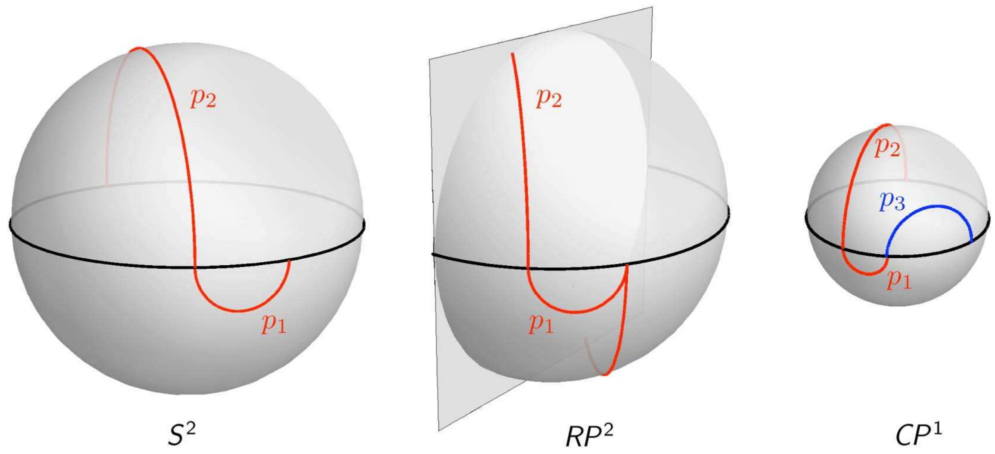

Figure 1. Two giant magnons are shown (in red) on the unit sphere S^2 (left), on RP^2 (centre, drawn here as half a sphere) and on CP^1, a sphere of radius $\frac{1}{2}$ (right). In all cases they have $p_1 = \frac{1}{2}$ and $p_2 = \pi - \frac{1}{2}$, which leads to a closed string in the RP^2 case, but not in the S^2 or CP^1 cases. In both the RP^2 and CP^1 cases, the equator is of length π, and we parameterise it by $\beta \in [0, \pi]$. The magnon with $p_1 = \frac{1}{2}$ spans $\Delta\beta = \frac{1}{2}$ in the RP^2 case, but only $\Delta\beta = \frac{1}{4}$ in the CP^1 case. On CP^1 we have also drawn a third magnon (in blue) with $p_3 = 1$, which spans the same length of equator $\Delta\beta = \frac{1}{2}$ as does the p_1 magnon on RP^2.

Figure 1 shows two magnons on S^2 and then on RP^2, one with $p = \frac{1}{2}$ and another with $p = \pi - \frac{1}{2}$. In the RP^2 case they have opposite opening angles $\delta = \pm\frac{1}{2}$, thus form a single closed string, while in the S^2 case the total opening angle is π.

5.3 The subspace RP^3

In the $AdS_5 \times S^5$ case, Dorey's giant magnons with a second large angular momentum $J' \sim \sqrt{\lambda}$ allow one to see that the dispersion relation is $\Delta - J_{\text{sphere}} = \sqrt{J'^2 + \frac{\lambda}{\pi^2}\sin^2(p/2)}$ [31]. These necessarily live in S^3 rather than S^2. They are called dyonic magnons, and (embedding $S^3 \subset \mathbb{C}^2$) can be written

$$W_1 = e^{it}\left(\cos\frac{p}{2} + i\sin\frac{p}{2}\,\tanh U\right)$$
$$W_2 = e^{iV}\sin\frac{p}{2}\,\text{sech}\,U$$

where

$$U = (x\cosh\beta - t\sinh\beta)\cos\alpha \qquad \cot\alpha = \frac{2r}{1-r^2}\sin\frac{p}{2}$$
$$V = (t\cosh\beta - x\sinh\beta)\sin\alpha \qquad \tanh\beta = \frac{2r}{1+r^2}\cos\frac{p}{2}\,.$$

The parameter p is still the opening angle along the equator in the W_1 plane, although $\cos(p/2)$ is clearly no longer the worldsheet velocity. Sending the new parameter $r \to 1$ reproduces the original giant magnon.

The second method of embedding S^2 solutions into CP^3, given by (5.8), points out a way to embed S^3 solutions:

$$\mathbf{Z} = \frac{1}{\sqrt{2}} \left(W_1, W_2, \bar{W}_2, \bar{W}_1 \right) . \tag{5.11}$$

As before, this is in fact a subspace RP^3 rather than S^3, thanks to the identification of $(w_1, w_2) \sim -(w_1, w_2)$ implied.[14]

Embedding a dyonic giant magnon in this way gives a CP^3 solution with charges[15]

$$\begin{aligned} \Delta - \frac{J_1 - J_4}{2} &= 2\sqrt{2\lambda} \frac{1 + r^2}{2r} \sin\left(\frac{p}{2}\right) \\ \frac{J_2 - J_3}{2} &= 2\sqrt{2\lambda} \frac{1 - r^2}{2r} \sin\left(\frac{p}{2}\right) . \end{aligned}$$

These satisfy the relation

$$\Delta - \frac{J_1 - J_4}{2} = \sqrt{\left(\frac{J_2 - J_3}{2}\right)^2 + 8\lambda \sin^2\left(\frac{p}{2}\right)}.$$

Notice that the second angular momentum here is that carried by Y^2 and $Y_3^\dagger$, which are the impurities we insert into the vacuum (2.2) to make magnons in the SU(2) $\times$ SU(2) sector.

This subspace can also be obtained from (3.4), by fixing $\vartheta_1 = \frac{\pi}{2}$, $\vartheta_2 = \frac{\pi}{2}$ and $\eta = 0$. The metric becomes

$$ds^2 = d\xi^2 + \sin^2\xi \, d\left(\frac{\varphi_2}{2}\right)^2 + \cos^2\xi \, d\left(\frac{\varphi_1}{2}\right)^2 .$$

This restriction can be checked to be a legal one from the equations of motion for the angles ϑ_1, ϑ_2 and η. The dyonic giant magnon in this space was re-derived by [17], using exactly these angles. It was also re-derived by [16] using co-ordinates $\mathbf{z}$.

Like the RP^2 magnons above, at $p = \pi$ these form single closed strings, and beyond this ($\pi < p < 2\pi$) give a second class of magnons connecting the same two points on the equator as the magnon with $\tilde{p} = p - \pi$.

6 Some larger subspaces

All of the solutions we have discussed so far are known from the $AdS_5 \times S^5$ case, and explore only subspaces S^2 or $S^3 \subset S^5$. In this section look at two subspaces of CP^3 on which new solutions might exist: CP^2 and $S^2 \times S^2$.

We also study restrictions of this $S^2 \times S^2$ down to three or two dimensions (in sections 6.3 and 6.4) since the resulting spaces have been used in the literature.

[14] Note that the rotation from $\mathbf{z}$ to $\mathbf{w}$ given by (5.9) is not an isometry, and in particular that the identification $\mathbf{z} \sim \lambda\mathbf{z}$ which defines CP^3 does not apply afterwards: $\mathbf{w} \nsim \lambda\mathbf{w}$ for complex λ. If $w_3 = w_4 = 0$, as is implied by (5.11), then the phases of w_1 and w_2 are both physical. (Which is good if we're claiming that the dyonic magnon has momenum along both of them).

However, the relation $\mathbf{w} \sim \lambda\mathbf{w}$ is true for real λ, and since we have fixed $w_1^2 + w_2^2 = 1$ by starting with a string solution on S^2, the identification $(w_1, w_2) \sim -(w_1, w_2)$ is all that survives.

[15] In calculating these charges from (4.5), the same cancellation of the second term happens here as happened in the previous section. Thus using the charges one would expect for $S^7 \subset \mathbb{C}^4$ gives the right answer here. This does not work in the CP^1 case, see footnote 10.

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6.1 The subspace CP^2

The first larger nontrivial subspace we can find is CP^2, obtained by setting $z_3 = 0$. In terms of the angles (3.4), the restriction is $\vartheta_2 = 0$ (and $\varphi_2 = 0$, since this is now redundant) and the metric becomes

$$ds^2 = d\xi^2 + \frac{1}{4}\cos^2\xi\left(d\vartheta_1^2 + \sin^2\vartheta_1\, d\varphi_1^2\right) + \frac{1}{4}\sin^2 2\xi\left(d\eta + \frac{1}{2}\cos\vartheta_1\, d\varphi_1\right)^2 .$$

The two manifest isometries here are along φ_1 and η. When $\xi = 0$ this is an S^2 equivalent to (5.4) (exchange $z_2 \leftrightarrow z_4$ to align them perfectly). Perhaps allowing $\xi \neq 0$ will allow new dyonic solutions here, generalising the CP^1 solution (5.6) just as the dyonic RP^3 solution generalises the RP^2 solution.

Note that this is certainly a legal subspace, for the same reason as given for CP^1: setting $z_3 = 0$ certainly solves the z_3 equation of motion.

6.2 The subspace $S^2 \times S^2$

If we set $\varphi_1 = \varphi_2$ and $\vartheta_1 = \vartheta_2$ in metric (3.4), we get the four-dimensional space

$$ds^2 = \frac{1}{4}\left[d(2\xi)^2 + \sin^2(2\xi)\, d\eta^2\right] + \frac{1}{4}\left[d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right] \tag{6.1}$$

which is $S^2 \times S^2$ (possibly up to co-ordinate ranges), and of course the new angles are defined $\vartheta \equiv (\vartheta_1 + \vartheta_2)/2$ and $\varphi \equiv (\varphi_1 + \varphi_2)/2$.

On such a product space, the Polyakov action splits into two terms, giving two non-interacting sets of target-space co-ordinates. Any two S^2 string solutions can be placed onto the same worldsheet, completely independently. Choosing giant magnon solutions, worldsheet scattering between these sectors would be trivial, just as it would be on two decoupled Heisenberg spin chains.

The restrictions needed to obtain this space are that $\vartheta_- \equiv \vartheta_1 - \vartheta_2 = 0$ and $\varphi_- \equiv \varphi_1 - \varphi_2 = 0$, and unfortunately the equations of motion for ϑ_- and φ_- are not automatically solved by this choice: instead they give complicated relations between the other co-ordinates. The equation for ϑ_- reads

$$\begin{aligned} 0 =& -\partial_t\left(\cos 2\xi\, \partial_t\vartheta\right) + \partial_x\left(\cos 2\xi\, \partial_x\vartheta\right) + \frac{1}{2}\cos 2\xi \sin 2\vartheta\left(\partial_t^2\varphi - \partial_x^2\varphi\right) \\ & - \sin^2 2\xi \sin\vartheta\left(\partial_t\eta\, \partial_t\varphi - \partial_x\eta\, \partial_x\varphi\right) \end{aligned}$$

and that for φ_- reads

$$\begin{aligned} 0 =& -\partial_t\left(\sin^2 2\xi \cos\vartheta\, \partial_t\eta + \cos 2\xi \sin^2\vartheta\, \partial_t\varphi\right) \\ & + \partial_x\left(\sin^2 2\xi \cos\vartheta\, \partial_x\eta + \cos 2\xi \sin^2\vartheta\, \partial_x\varphi\right) . \end{aligned}$$

These constraints do not of course rule out the existence of solutions on this subspace. But placing an arbitrary S^2 solution into each of the factors is unlikely to produce a solution, because of these equations coupling ξ, η to ϑ, φ.

6.3 The subspace $S^2 \times S^1$

If we further restrict the above subspace by holding one of the angles fixed, we will get $S^2 \times S^1$ (again up to identifications). Setting $\vartheta = \frac{\pi}{2}$ gives the space studied by [17], with metric

$$ds^2 = \frac{1}{4}\left[d(2\xi)^2 + \sin^2(2\xi)\, d\eta^2 + d\varphi^2\right].$$

The equation of motion for ϑ is solved by $\vartheta = \frac{\pi}{2}$, and the constraints imposed by $\vartheta_- = 0$ and $\varphi_- = 0$ above simplify to

$$0 = -\partial_t\eta\, \partial_t\varphi + \partial_x\eta\, \partial_x\varphi \tag{6.2}$$

$$0 = -\partial_t\left(\cos 2\xi\, \partial_t\varphi\right) + \partial_x\left(\cos 2\xi\, \partial_x\varphi\right). \tag{6.3}$$

These constraints were not taken into account by [17], who sets $\vartheta_- = 0$ before calculating the equation of motion for ϑ (which is indeed solved) but without ever calculating the equation of motion for ϑ_-.[16] The magnon ansatz used there sets $\eta = \omega t + f(u)$, $\varphi = \nu t$ and $\xi = g(u)$, in terms of boosted $u = \beta t + \alpha x$. The first constraint (6.2) then implies $\beta\, f'(u) = -\omega$, while for a magnon solution one typically has $f(u) \propto \tanh u$. The second constraint (6.3) implies $\beta = 0$, so together they imply $\omega = 0$.

This problem does not arise in the other case studied by [17], where the ϑ_- equation is solved by $\eta = 0$, and $\varphi_1 \neq \varphi_2$ so there is no φ_- constraint. The resulting subspace is the RP^3 discussed in section 5.3.

6.4 The subspace CP^1, again

Finally, we can restrict the subspace $S^2 \times S^2$ of (6.1) by holding both of the angles in one factor constant, to obtain S^2. Setting ξ and η to be constants leaves the space

$$ds^2 = \frac{1}{4}\left[d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right]$$

which is, like our CP^1 of section 5.1, a sphere of radius $\frac{1}{2}$. This is a legal subspace, as the equations of motion for ξ and η are automatically solved (because a stationary particle anywhere on the sphere is a solution) and the constraints arising from $\vartheta_- = 0$ and from $\varphi_- = 0$ become simply the equations of motion for ϑ and φ.

When $\xi = \frac{\pi}{2}$, and using the conventions given in appendix A, this space is embedded by

$$\mathbf{z} = \left(e^{i\varphi/2}\cos\frac{\vartheta}{2},\ 0,\ 0,\ e^{-i\varphi/2}\sin\frac{\vartheta}{2}\right).$$

This is precisely the same subspace CP^1 as in (5.4), although we obtained it there by fixing $\alpha = 0$ in the other set of angles (3.3). Fixing ξ to some other value will simply rotate the 1-2 and 3-4 planes, but in all cases the space is $S^2 = CP^1$. Like the subspace RP^2 discussed in section 5.2, this one is sometimes referred to as $S^2 \times S^2$ in the literature.

These co-ordinates were used by [32] to study finite-J effects on the CP^1 giant magnon. We give their results in (7.2) below.

[16] The constraint (6.2) can also be obtained without using ϑ_-, by simply setting $\vartheta_1 = \frac{\pi}{2}$ and $\vartheta_2 = \frac{\pi}{2}$ in their equations of motion.

7 Finite-J corrections

All of the giant magnons we have written down so far have both infinite energy and infinite angular momentum. As can be seen from (5.2), this corresponds to infinite worldsheet length in the timelike conformal gauge we are using.

The first treatment of giant magnons $AdS_5 \times S^5$ at finite J was by [33], who worked in uniform lightcone gauge, in which the worldsheet density of J, rather than of Δ, is constant. Their gauge has a parameter $a \in [0,1]$, and at $a=0$ (and in conformal gauge) they obtained the following correction to the dispersion relation:

$$\begin{aligned}\varepsilon \equiv \Delta - J &= \frac{\sqrt{\lambda}}{\pi} \sin\left(\frac{p}{2}\right) \left[1 - \frac{4}{e^2} \sin^2\left(\frac{p}{2}\right) e^{-2J/\varepsilon} + o(e^{-4J/\varepsilon})\right] \\ &= \frac{\sqrt{\lambda}}{\pi} \sin\left(\frac{p}{2}\right) \left[1 - 4\sin^2\left(\frac{p}{2}\right) e^{-2\Delta/\varepsilon} + \cdots\right]\end{aligned}$$

Exact solutions at any J were studied by [34], where it was shown that they are connected by the Pohlmeyer map to kink-train solutions of sine-gordon theory. The apparent gauge-dependence of the results of [33] was resolved by [35], using the fact that the solutions are periodic both on the worldsheet and in the azimuthal angle on the sphere, and so can be viewed as wound strings on $S^2/\mathbb{Z}_n$ [35, 36]. The scattering of finite-J magnons was studied by [37], using the connection to sine-gordon theory in finite volume.

The finite-J generalisations of the basic giant magnon are still solutions moving on S^2, and so one can place them into CP^3 using either of the maps presented in sections 5.1 and 5.2 above. For the RP^2 giant magnon, the corrected dispersion relation was derived by [38] to be

$$\Delta - \frac{J_1 - J_4}{2} = 2\sqrt{2\lambda} \sin\left(\frac{p}{2}\right) \left[1 - 4\sin^2\left(\frac{p}{2}\right) e^{-2\Delta \big/ 2\sqrt{2\lambda}\sin(\frac{p}{2})} + \cdots\right]. \tag{7.1}$$

For the CP^1 giant magnon, [32] give the result[17]

$$\Delta - \frac{J_1 - J_4}{2} = \sqrt{2\lambda} \sin\left(\frac{p}{2}\right) \left[1 - 4\sin^2\left(\frac{p}{2}\right) e^{-2\Delta \big/ \sqrt{2\lambda}\sin(\frac{p}{2})} + \cdots\right]. \tag{7.2}$$

[17] Here is brief note about deriving these two results from the original S^2 case. The integrals defining the charges are now over a finite length $-L < x < L$, so write $J(L)$ and $\Delta(L)$. Note that $\Delta(2L) = 2\Delta(L)$. To get the charges for one magnon, we must integrate from one cusp to the next: choose L such that $\theta_{\text{mag}}(x = \pm L, t = 0)$ are at the first cusps.

For the RP^2 case, the relationship we used before $J_{\text{sphere}}(L) = J_1(L) = (J_1(L) - J_4(L))/2$ still holds, leading to (7.1). We wrote the S^2 result above using the prefactor appropriate for $AdS_5 \times S^5$, so to get this result for the $AdS_4 \times CP^3$ theory have replaced $\sqrt{\lambda}/\pi \to 2\sqrt{2\lambda}$.

For the CP^1 case, the cusp at $\theta_{\text{mag}}(L,0)$ is at $\mathbf{Z}_{CP^1}(\frac{L}{2}, 0)$, thanks to the scaling (5.5). The relationship between charges is that

$$\frac{J_1(\frac{L}{2}) - J_4(\frac{L}{2})}{2} = \frac{1}{2} J_{\text{sphere}}(L).$$

Thus $\Delta(\frac{L}{2}) - (J_1(\frac{L}{2}) - J_4(\frac{L}{2}))/2 = \Delta(\frac{L}{2}) - \frac{1}{2} J_{\text{sphere}}(L) = \frac{1}{2}(\Delta(L) - J_{\text{sphere}}(L))$. In the result (7.2), it is the energy for one magnon $\Delta(\frac{L}{2})$ which appears both on the left hand side and in the exponent.

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We observe that, even at finite J, two CP^1 magnons have the same dispersion relation as one RP^2 magnon, provided all three have the same value of the parameter p.[18]

Dyonic giant magnons can also be studied at finite J; this has been done for those in S^5 from this string sigma-model perspective by [34, 39], and for those in $RP^3 \subset CP^3$ by [16, 40].

In the $AdS_5 \times S^5$ case these corrections can also be calculated using algebraic curves [41] or using the Lüscher formula [42], and these agree with the string sigma-model result presented above. For calculations on the gauge theory side of the correspondance see [43]. In $AdS_4 \times CP^3$ the same list of methods is possible, and we discuss these further in section 8.3 below.

8 Discussion and conclusion

In this paper we have only discussed giant magnon solutions known from $AdS_5 \times S^5$, but have been careful about how these are placed into CP^3. Here we summarise these results, comment on more general solutions, and comment on connections to approaches other than the classical string sigma-model.

8.1 Single-charge giant magnons

In sections 5.1 and 5.2 we looked at two different ways to embed the basic single-charge giant magnon (5.1), into either CP^1 or RP^2 [10, 11]. This CP^1 is a two-sphere of radius $\frac{1}{2}$, while RP^2 is half a two-sphere, so both have an equator of length π. We lined up the embeddings into $\mathbb{C}^4$ such that, in both cases, the equator is the line

$$\mathbf{z} = \frac{1}{\sqrt{2}}\left(e^{i\beta}, 0, 0, e^{-i\beta}\right)$$

where we name the angle $\beta \in [0, \pi]$, as in (4.2) above, to avoid confusion.

Since the basic magnon (5.1) has opening angle $\Delta\phi_{\rm mag} = p$, these two solutions have

$$\begin{array}{llll} CP^1: & \beta = \chi/4 = \phi_{\rm mag}/2 & \Longrightarrow & \Delta\beta = p/2 \\ RP^2: & \beta = \varphi_2/2 = \phi_{\rm mag} & \Longrightarrow & \Delta\beta = p' \end{array}$$

(where we now write p' for the parameter of the RP^2 magnon, to distinguish it from the CP^1 case's p). A single giant magnon is not a closed string solution, one must join a set of them together at their endpoints on the equator. The condition for a set p_i of CP^1 magnons or p'_j of RP^2 magnons to close is that the total opening angle $\Delta\beta$ should be a multiple of π, that is,

$$\begin{array}{lll} CP^1: & \displaystyle\sum_i p_i = 2\pi n & \\ RP^2: & \displaystyle\sum_j 2p'_j = 2\pi n\,, & n \in \mathbb{Z}. \end{array} \tag{8.1}$$

[18] Note that that essentially all the properties of the two CP^1 magnons add up to give those of the single RP^2 magnon: energy Δ, angular momentum $(J_1 - J_4)/2$, worldsheet length L and opening angle along the equator (which we call $\Delta\beta$ in the next section).

The point particle (3.5) moves along the same equator too, and by calculating fluctuations of this solution, we checked in section 4 that $\Delta - \frac{J_1-J_4}{2}$ is indeed a Hamiltonian for them, just as $\Delta - J$ is in the S^5 case. Calculating the same difference of charges for the two magnon embeddings, we obtained dispersion relations (5.7) and (5.10), which we now write also in terms of the opening angle $\Delta\beta$:

$$\begin{aligned} CP^1: \qquad & \Delta - \frac{J_1-J_4}{2} = \sqrt{2\lambda}\sin\left(\frac{p}{2}\right) \quad = \sqrt{2\lambda}\sin\left(\Delta\beta\right) \\ RP^2: \qquad & \Delta - \frac{J_1-J_4}{2} = 2\sqrt{2\lambda}\sin\left(\frac{p'}{2}\right) = 2\sqrt{2\lambda}\sin\left(\frac{\Delta\beta}{2}\right). \end{aligned}$$

Notice that these agree at small $\Delta\beta$. The limit $p \to 0$ takes you from giant magnons to the Penrose limit (via the interpolating case of [44], studied here by [45]). Finite-J effects in the Penrose limit were studied by [46].

As noted in section 5.2, there is also a second magnon on RP^2 for any given opening angle $\Delta\beta$, which has charges [10]

$$RP^{2\prime}: \qquad \Delta - \frac{J_1-J_4}{2} = 2\sqrt{2\lambda}\cos\left(\frac{\Delta\beta}{2}\right).$$

For small $\Delta\beta$ this is almost a circular string, with its ends slightly offset along the equator — see figure 1 on page 11 above.

8.2 More solutions!

While we used the giant magnon on S^2 (5.1) as an example, the subspaces we have described exist independently of it, and any other string solution moving on S^2 can be placed into either of these subspaces of CP^3 in the same way. Thus not only finite-J magnons (as discussed in section 7 above) but also scattering solutions [47] and single spikes[19] [48–50] all exist in both the CP^1 and RP^2 subspaces. The equations of motion do not notice the global identification $(w_1, w_2) \sim -(w_1, w_2)$ which distinguishes RP^2 from S^2, and the fact that CP^1 is a sphere of radius $\frac{1}{2}$ can be dealt with by the same scaling (5.5) that we used for the basic magnon.

Many papers interpret the magnon on RP^2 (and also that on RP^3) as being two magnons, one in each half of the embedding space $\mathbb{C}^2 \times \mathbb{C}^2$ [11, 16]. It is then tempting to identify these two halves with the even- and odd-site spin chains in the dual description's $\mathrm{SU}(2)\times\mathrm{SU}(2)$ sector. For the known solutions, however, these two halves are not independent: in fact they are always locked together, and by a trivial change of co-ordinates (5.9) we can write them as a single $RP^2 = S^2/\mathbb{Z}_2$ space. This does not rule out the existence of two independent magnon sectors, such that a pair of magnons of the same parameter p gives us again the known RP^2 solution. But at present individual solutions in these two sectors are not known.

[19] Single-spike solutions of all kinds can be easily obtained from their giant magnon partners by the $x \leftrightarrow t$ exchange discussed in [48, 49]. As in $\mathbb{R}\times S^5$, this exchange (keeping $X^0 = t$) is a symmetry of the equations of motion (B.2) and the Virasoro constraints for $\mathbb{R}\times CP^3$. Thus the classical solutions have no properties which cannot be read off from the corresponding magnon solution. However, the quantum properties are quite different [49].

The single-parameter giant magnon on S^2 has a two-parameter dyonic generalisation on S^3, and in section 5.3 we looked at how to map this into $RP^3 \subset CP^3$, where it generalises the RP^2 solution. The dyonic generalisation of the CP^1 solution is not known, but it might lie in the CP^2 subspace we discussed in section 6.1.

It would be very interesting to find some indication among the magnon solutions of the weaker momentum constraint: the momentum in just the even-site or just the odd-site spin chain need not vanish, only the total. Combining the two closure conditions (8.1) to give $\sum_i p_i + \sum_j 2p'_j = 2\pi n$ cannot be the answer, because these two classes of magnons are certainly inequivalent solutions, while the even- and odd-site spin chains are related by an SU(4) rotation.

8.3 Beyond the classical sigma-model

The classical string solutions we have discussed are well-known from the S^5 case, and explore only S^2 or S^3-like subspaces of CP^3. Their classical properties (and indeed those of solutions we have not discussed, such as scattering solutions) are not strongly affected by being transplanted to the new space. However, their quantum properties will certainly depend on the whole space, as was the case for spinning string solutions in $AdS_2 \times S^1$ studied by [5]. The relevant supersymmetric sigma model (for strings on $AdS_4 \times CP^3$) was first studied by [6, 7]. Using this one would like to perform a calculation like that done for magnons in $AdS_5 \times S^5$ by [30].

Like the equations of motion, the Pohlmeyer map [51] to the sine-gordon field α (given by $\cos\alpha = -\partial_t \bar{W}_i \partial_t W_i + \partial_x \bar{W}_i \partial_x W_i$ in the S^2 case) depends only locally on the target-space co-ordinates. Thus strings on either CP^1 or RP^2 will be classically equivalent to the sine-gordon model. The condition that the string closes $\sum \Delta\beta \sim 0$ plays no role in the sine-gordon model, thus the second class of magnons, which we called $RP^{2\prime}$ above, has no special meaning in sine-gordon theory. As quantum systems, strings on $\mathbb{R} \times S^2$ are quite different to the sine-gordon model, thanks to the different notion of energy, and this complicates the translation of the n-body description of solitons in sine-gordon theory to this case [3, 52, 53]. The Pohlmeyer reduction has been extended to the full superstring on $AdS_5 \times S^5$, [54] and also to strings moving on CP^3 [55].

Classical strings in $AdS_4 \times CP^3$ can also be studied using the algebraic curve, in which the 10 eigenvalues q_a of the monodromy matrix Ω are analytic functions of the spectral parameter, and their various poles and branch points control the solution [56]. Giant magnons in this picture were studied by [57], and are of two distinct kinds, 'small' and 'big'. Their dispersion relations are as follows:

$$\begin{aligned}
&\text{small GM:} \qquad \varepsilon = \sqrt{\frac{1}{4} + 2\lambda \sin^2\left(\frac{p}{2}\right)} \quad \to \sqrt{2\lambda}\,\sin\left(\frac{p}{2}\right) \qquad \text{when } \sqrt{\lambda} \gg 1 \\
&\text{big GM:} \qquad \varepsilon = \sqrt{1 + 8\lambda \sin^2\left(\frac{p}{4}\right)} \quad \to 2\sqrt{2\lambda}\,\sin\left(\frac{p}{4}\right).
\end{aligned}$$

It would seem natural to identify these with the CP^1 and RP^2 magnons of the string sigma-model, presumably with $p' = p/2 = \Delta\beta$. There are two 'small GM' sectors, together often called the SU(2) × SU(2) sector.

However, the study of finite-J corrections to these paints a different picture. According to [58], two 'small GM's in the two sectors, both with the same momentum p, have a correction $\delta\varepsilon$ matching the RP^2 string result (7.1). This does seems to point to the interpretation of the RP^2 string solution as two giant magnons, as was originally claimed by [11]. However, the same paper's result for one 'small GM' does not match any of the string calculations, apparently leaving open the identification both of the string state for this, and of the algebraic curve corresponding to the CP^1 string. Finite-J corrections have also been studied using the Lüscher formula by [58, 59], and the results agree with those from the algebraic curve.

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A More about CP^3's geometry

The complex projective space CP^3 is defined to be

$$CP^3 = \frac{\mathbb{C}^4}{\mathbf{z} \sim \lambda \mathbf{z}}$$

where $\mathbf{z} = z_a$ are called homogeneous co-ordinates. We can split this identification into $\mathbf{z} \sim r\mathbf{z}$ and $\mathbf{z} \sim e^{i\phi}\mathbf{z}$ (for any $r, \phi \in \mathbb{R}$) and then replace the first one with the condition $|\mathbf{z}|^2 = 1$, to obtain a sphere with one identification

$$CP^3 = \frac{S^7}{\mathbf{z} \sim e^{i\phi}\mathbf{z}} = \frac{S^7}{\mathrm{U}(1)}\,.$$

The isometry group is SU(4), acting in the natural way on $\mathbf{z}$. Since the stabiliser group of (say) the point $z_4 = 1$ is U(3), we can also write

$$CP^3 = \frac{\mathrm{SU}(4)}{\mathrm{U}(3)}\,.$$

The infinitesimal form of the standard Fubini-Study metric for this is

$$\begin{aligned} ds^2_{CP^3} &= \frac{dz_i d\bar{z}_i}{\rho^2} - \frac{|z_i d\bar{z}_i|^2}{\rho^4} \\ &= ds^2_{\text{sphere}} - d\gamma^2 \\ &= \frac{ds^2_{\text{flat}} - d\rho^2}{\rho^2} - d\gamma^2 \end{aligned} \tag{A.1}$$

where $\rho^2 = z_i \bar{z}_i$. (Note that in some conventions the metric is 4 times this, [10, 60] making CP^1 (5.4) a unit sphere). In the second and third lines above, $ds^2_{\text{flat}} = dz_i d\bar{z}_i$ is the Euclidean metric for $\mathbb{C}^4$, and ds^2_{sphere} is a metric for S^7 in terms of these embedding co-ordinates. Instead of fixing $\rho = 1$, this way of treating the sphere subtracts off the component coming from radial motion (and scales the rest appropriately). In turn, CP^3 can be obtained from the sphere by fixing the total phase $\gamma = \arg \prod_i z_i$, or instead by subtracting the total phase component. These two pieces are

$$d\rho = \frac{1}{2\rho}\left(z_i d\bar{z}_i + \bar{z}_i dz_i\right) = \frac{1}{\rho}\,\text{Re}\left(\bar{z}_i dz_i\right)$$
$$d\gamma = \frac{i}{2\rho^2}\left(z_i d\bar{z}_i - \bar{z}_i dz_i\right) = \frac{1}{\rho^2}\,\text{Im}\left(\bar{z}_i dz_i\right).$$

We now present the maps between the homogeneous co-ordinates and the two sets of angles we have used. These are taken from [29] and [25], although we have shuffled the z_i. For the metric (3.4) (whose η is often called ψ)

$$\begin{aligned} ds^2_{CP^3} = d\xi^2 &+ \frac{1}{4}\sin^2 2\xi \left(d\eta + \frac{1}{2}\cos\vartheta_1\, d\varphi_1 - \frac{1}{2}\cos\vartheta_2\, d\varphi_2\right)^2 \\ &+ \frac{1}{4}\cos^2\xi\left(d\vartheta_1^2 + \sin^2\vartheta_1\, d\varphi_1^2\right) + \frac{1}{4}\sin^2\xi\left(d\vartheta_2^2 + \sin^2\vartheta_2\, d\varphi_2^2\right) \end{aligned}$$

the relationship is:

$$\begin{aligned} z_1 &= \sin\xi\ \cos(\vartheta_2/2)\, e^{-i\eta/2}\, e^{i\varphi_2/2} \\ z_2 &= \cos\xi\ \cos(\vartheta_1/2)\, e^{i\eta/2}\, e^{i\varphi_1/2} \\ z_3 &= \cos\xi\ \sin(\vartheta_1/2)\, e^{i\eta/2}\, e^{-i\varphi_1/2} \\ z_4 &= \sin\xi\ \sin(\vartheta_2/2)\, e^{-i\eta/2}\, e^{-i\varphi_2/2}. \end{aligned} \tag{A.2}$$

For the other set of angular variables (3.3)

$$\begin{aligned} ds^2_{CP^3} = d\mu^2 &+ \frac{1}{4}\sin^2\mu\cos^2\mu\left[d\chi + \sin^2\alpha\ (d\psi + \cos\theta\ d\phi)\right]^2 \\ &+ \sin^2\mu\left[d\alpha^2 + \frac{1}{4}\sin^2\alpha\left(d\theta^2 + \sin^2\theta\ d\phi^2 + \cos^2\alpha\,(d\psi + \cos\theta\ d\phi)^2\right)\right] \end{aligned}$$

the map is specified by

$$\begin{aligned} z_1/z_4 &= \tan\mu\ \cos\alpha\ e^{i\chi/2} \\ z_2/z_4 &= \tan\mu\ \sin\alpha\ \sin(\theta/2)\ e^{i\chi/2}\ e^{i(\psi-\phi)/2} \\ z_3/z_4 &= \tan\mu\ \cos\alpha\ \cos(\theta/2)\ e^{i\chi/2}\ e^{i(\psi+\phi)/2}. \end{aligned} \tag{A.3}$$

These ratios z_i/z_4 are called inhomogeneous co-ordinates, and cover the patch $z_4 \neq 0$ with no identifications [60]. With the ranges given, the trigonometric functions controlling the amplitudes are always positive in both of these cases. From the phases of the inhomogeneous co-ordinates of z_i/z_4 it is easy to see that ranges of the remaining angles are correct.

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B Strings in homogeneous co-ordinates

To study bosonic string theory in S^n, it is often convenient to use embedding co-ordinates for $\mathbb{R}^{n+1}$ and then constrain the radius to 1. This avoids all the trigonometric functions needed for angular co-ordinates, and (in AdS/CFT) also gives a simple correspondence between the R-symmetry generators and the rotations of this space. We can do the same for CP^3, using homogeneous co-ordinates $\mathbf{z}$. We will need two constraints, $\rho^2 = 1$ and $\gamma = 0$.

B.1 Using Lagrange multipliers

Begin by writing the metric for $\mathbb{R} \times CP^3$ as

$$ds^2 = -\left(dX^0\right)^2 + d\bar{z}_i G_{ij} dz_j \qquad \text{with } \; G_{ij} = \frac{\delta_{ij}}{\rho^2} - \frac{z_i \bar{z}_j}{\rho^4}$$

In conformal gauge, and with $X^0 = \kappa t$, the Polyakov action is

$$\begin{aligned} S &= \int \frac{dx\, dt}{2\pi} R^2 \mathcal{L} \qquad\qquad (B.1) \\ &= 2\sqrt{2\lambda} \int dx\, dt\, \mathcal{L} \\ 2\mathcal{L} &= \kappa^2 + \partial^a \bar{Z}_i G_{ij} \partial_a Z_j + \Lambda_\rho \left(\bar{Z}_i Z_i - 1\right) + i\Lambda_\gamma \left(Z_1 Z_2 Z_3 Z_4 - \bar{Z}_1 \bar{Z}_2 \bar{Z}_3 \bar{Z}_4\right). \end{aligned}$$

Note that $\Lambda_\gamma \in \mathbb{R}$, since the piece in brackets is proportional to $2i \sin\gamma$. In calculating Euler-Lagrange equations for this, we set $\rho = 1$ immediately, simplifying $\partial G_{ij}/\partial Z_i$ etc. greatly. The Lagrange multipliers can be read off from the parallel component of the equations (i.e. $\bar{Z}_i$ times Z_i's equation of motion) which is:

$$\Lambda_\rho - 4i \left(Z_1 Z_2 Z_3 Z_4\right) \Lambda_\gamma = \partial_t \bar{Z}_i \partial_t Z_i - 2 \left|\bar{Z}_i \partial_t Z_i\right|^2 \; - \; \partial_x \bar{Z}_i \partial_x Z_i + 2 \left|\bar{Z}_i \partial_x Z_i\right|^2.$$

(This 4 is the number of complex embedding co-ordinates). The right-hand side here is real, which implies $\Lambda_\gamma = 0$. Using this, we find the equation of motion for Z_i to be

$$-\partial_t \left(G_{ij} \partial_t Z_j\right) + \partial_x \left(G_{ij} \partial_x Z_j\right) = Z_i \Lambda_\rho - \left(\bar{Z}_j \partial_t Z_j\right) \partial_t Z_i + \left(\bar{Z}_j \partial_x Z_j\right) \partial_x Z_i\,. \qquad (B.2)$$

The Virasoro constraints are

$$\begin{aligned} -\kappa^2 + \partial_t \bar{Z}_i\, G_{ij}\, \partial_t Z_j + \partial_x \bar{Z}_i\, G_{ij}\, \partial_x Z_j &= 0 \\ \operatorname{Re}\left(\partial_t \bar{Z}_i\, G_{ij}\, \partial_x Z_j\right) &= 0\,. \end{aligned}$$

The result that $\Lambda_\gamma = 0$ deserves a little explanation. If we were to analyse strings on the sphere using a similar metric (in fact exactly ds^2_{sphere} from (A.1) above):

$$2\mathcal{L} = 1 + \partial^a X_i \partial_a X_j g_{ij} + \Lambda(X^2 - 1), \qquad \text{with } \; g_{ij} = \frac{\delta_{ij}}{\rho^2} - \frac{X_i X_j}{\rho^4}$$

then we would also find $\Lambda = 0$, although the equations of motion are the same as are obtained with $g_{ij} = \delta_{ij}$ (i.e. using ds^2_{flat}). In some sense the metric is enforcing the constraint for us. The reason we had $\Lambda_\rho \neq 0$ in the CP^3 case above was that we set $\rho = 1$ at an early stage of the calculation.

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B.2 Constraining S^7 solutions

The approach of [11] (and others) to strings on CP^3 is to find solutions on the sphere $S^7 \in \mathbb{C}^4$, and then further demand that the two Noether charges from ∂_γ vanish:

$$0 = C_0 \equiv \sum_{i=1}^{4} \mathrm{Im}\left(\bar{Z}_i \partial_t Z_i\right) , \qquad 0 = C_1 \equiv \sum_{i=1}^{4} \mathrm{Im}\left(\bar{Z}_i \partial_x Z_i\right) .$$

This is true for the RP^2 solution (5.8) given by [11], and more generally, for any solution on the larger RP^3 subspace of section 5.3. In terms of the co-ordinates $\mathbf{w}$ from (5.9), the condition $w_3 = w_4 = 0$ which defines this subspace implies $C_0 = C_1 = 0$, and also reduces the equations of motion (B.2) to those for the sphere S^3 embedded in (w_1, w_2).

But more general solutions, such as the CP^1 solution (5.6), do not solve these constraints, nor do they solve the equations of motion for $S^7 \subset \mathbb{C}^4$. So these conditions (solution on S^7, and $C_0 = C_1 = 0$) are certainly not necessary for a solution. Whether they are sufficient is not entirely clear to us.[20]

We noted in section 5.3 that when working in the subspace RP^3, the second term in the definition of charges J_i (4.5) vanishes, and what is left is the definition of the conserved charge from rotational symmetry of the z_i plane one would expect in S^7. Here we can add that the term which vanishes is $|Z_i|^2 C_0/\rho^4$. This does not vanish for the CP^1 case (5.6), see footnote 10.

Finally, we note that in terms of charges J_i we used throughout, something like the constraint $C_0 = 0$ does hold: $\sum_{i=1}^4 J_i = 0$ follows trivially from the definition (4.5).

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[20] A similar approach to strings on the sphere is to find solutions in flat (embedding) space and then reject all those which do not have $\rho = 1$. In this case solving the flat space equations and having $\rho = 1$ is sufficient to find a solution, but not necessary.

For example, when studying loops of string rotating in S^3, there is one critical speed at which they are solutions in unconstrained $\mathbb{R}^4$ too [61]. But faster and slower motions are possible on the sphere, with extreme cases of a point particle and a stationary hoop, which are not solutions in $\mathbb{R}^4$.

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